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Well-Posedness and Stability for Semilinear Thermoelastic System with Boundary Time-Varying Delay and Nonlinear Weight

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Abstract. This paper is concerned with the well-posedness, uniform asymptotic stability and dynamics for a semilinear thermoelastic system with time-varying delay boundary feedback and nonlinear weight, which can be used to describe the physical procedure of meridian retraction and release therapy. The perturbation theory of linear operators by Kato is used to deal with the invalidity of Lumper-Phillips theorem on non-autonomous PDEs operator, the multiplier approach and quasi-stability method lead to the stability and dynamics for our semilinear problem, which are also true for linear thermoelastic system without weight.

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1 Introduction

The delay and memory influence the stability and dynamics for evolutionary differential equations, which come from physics, biology, medicine, material, artificial intelligence and applied science/engineer, such as the transmission problem of hyperbolic equations.

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In the traditional Chinese medicine, there exists a physical therapy procedure which is called as meridian retraction and release therapy. The principle of this therapy can be seen as an abstract and simplified mathematical model influenced by delay, which is consisted by a coupled system via wave propagation and heat transportation with time-varying delay boundary feedback and nonlinear weight as the following semilinear thermoelastic system

$$\begin{cases} au_{tt}-du_{xx}+\beta\theta_x+h_1(u)=0, & \text{in } (0,L)\times(0,\infty),\\ b\theta_t-\kappa\theta_{xx}+\beta u_{xt}+h_2(\theta)=0, & \text{in } (0,L)\times(0,\infty),\\ u_x(0,t)=u(L,t)=\theta(0,t)=0, & t\geq 0,\\ \theta_x(L,t)+k_1(t)\theta(L,t)+k_2(t)\theta(L,t-\tau(t))=0, & t\geq 0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \theta(x,0)=\theta_0(x), & t\geq 0,\\ \theta(L,t-\tau(0))=f_0(L,t-\tau(0)), & \text{in } (0,L)\times(0,\tau(0)), \end{cases} \tag{1.1}$$

where (u_0,u_1,θ_0,f_0) belongs to some appropriate Sobolev space, $u(t,x)$ is the displacement of wave along meridians, θ denotes the heat flow which obeys the Fourier law, $h_i(\cdot)$ $(i=1,2)$ are the semilinear external forces caused by the retraction and release therapy technique on the boundary point $x=L$. Since the stability and dynamics are important in curative effect from the view of theory, the discussion of problem (1.1) is our objective in this paper.

When the semilinear terms $h_1(u)$ and $h_2(\theta)$ equal to 0, the weights $k_1(t)$ and $k_2(t)$ reduce to constants, the system (1.1) is degraded into the problem in [1]. Especially, if the time-varying delay $\tau(t)$ becomes constant, (1.1) reduces to the problem in [2].

As our best acknowledge, the research on hyperbolic equation with delay has been investigated in fruitful literatures, for instance [3–13]. The thermoelastic systems contain three types according to the different damping, which are composed by wave equation and heat flow. The well-posedness and stability for classical thermoelastic system have been studied in last thirty years, which can be referred in monographs as Jiang and Racke [14], Liu and Zheng [15]. The controllability and stability for thermoelastic system can be seen in [16, 17] and some related literatures therein. Originated from the idea in Nicaise, Pignotti and Valein [8], Mustafa [1, 2] considered the thermoelastic systems with boundary feedbacks which contain constant, time-varying and distributed delays, and derived the stability of energy functional. Thereafter, Mustafa and his collaborator Kafini, Messaoudi in [18–20] investigated the Timoshenko-type system of thermoelasticity with delays and showed the energy decay. Based on the above related literatures, the global well-posedness, stability and dynamics for semilinear thermoelastic system with time-varying delay and nonlinear weight are our objective in this presented paper, which contains the following results and features.

(1) The problem (1.1) is using to describe the transportation in medicine here, the global well-posedness has been achieved by semigroup theory in [21] together with the