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Strong Instability of Standing Waves for a Type of Hartree Equations

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Abstract. In this paper, we study the following three-dimensional Schrödinger equation with combined Hartree-type and power-type nonlinearities

$$i\partial_t\psi+\Delta\psi+(|x|^{-2}*|\psi|^2)\psi+|\psi|^{p-1}\psi=0$$

with $1<p<5$. Using standard variational arguments, the existence of ground state solutions is obtained. And then we prove that when $p\geq 3$, the standing wave solution $e^{i\omega t}u_\omega(x)$ is strongly unstable for the frequency $\omega>0$.

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1 Introduction

Consider the following three-dimensional Schrödinger equation with combined Hartree-type and power-type nonlinearities

$$i\partial_t\psi+\Delta\psi+(|x|^{-2}*|\psi|^2)\psi+|\psi|^{p-1}\psi=0 \tag{1.1}$$

with $1<p<5$, where $\psi=\psi(t,x)$ is a complex-valued wave function in time-space $(t,x)\in\mathbb{R}\times\mathbb{R}^3$; i is the imaginary unit; Δ is the Laplace operator on $\mathbb{R}^3$; $*$ denotes the standard convolution of the integral kernal $|x|^{-2}$ and the square term $|\psi|^2$ on $\mathbb{R}^3$. More precisely,

$$(|x|^{-2}*|\psi|^2)(x)=\int_{\mathbb{R}^3}\frac{|\psi(y)|^2}{|x-y|^2}\mathrm{d}y.$$

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The term $(|x|^{-2}*|\psi|^2)\psi$ is called the Hartree-type (or the nonlocal nonlinearity) and $|\psi|^{p-1}\psi$ is called power-type (or the local nonlinearity).

(1.1) origins from the Schrödinger-Poisson-Slater system, which was introduced by Slater [1] in approximation of Hartree-Fock equations. In particular, (1.1) without the Hartree-type is a canonical nonlinear Schrödinger equation, which may model the Bose-Einstein condensate, see e.g., [2]. (1.1) without the power-type is a standard nonlinear Hartree equation, which can be considered as a classical limit of a field equation describing a quantum mechanical nonrelativeistic many-boson system interacting through a two body potential $|x|^{-2}$, see e.g., [3]. (1.1) can be viewed as a generalization of the Poission equation for the gravitational potential that is typical in the study of fractional Newtonian gravity as an alterative to standard Newtonian gravity, see e.g.,[4], which has been extensively studied over the last few years, such as [5-7] and the references therein.

For (1.1), the local well-posedness in the natural energy space $H^1(\mathbb{R}^3)$ is established in [8] (also see [9]). That is, for any initial data $\psi_0(x)\in H^1(\mathbb{R}^3)$, there exists a unique maximal solution

$$\psi(t)\in\mathcal{C}((-T_{\min},T_{\max}),H^1(\mathbb{R}^3))\cap\mathcal{C}^1((-T_{\min},T_{\max}),H^{-1}(\mathbb{R}^3))$$

of (1.1) with $\psi(0)=\psi_0$. In addition, the mass

$$M(\psi(t)):=\|\psi(t)\|_2^2$$

and energy

$$E(\psi(t)):=\frac{1}{2}\|\nabla\psi(t)\|_2^2-\frac{1}{4}\|(|x|^{-2}*|\psi(t)|^2)|\psi(t)|^2\|_1-\frac{1}{p+1}\|\psi(t)\|_{p+1}^{p+1}$$

of the solution are conserved by flow. Moreover, there holds the blow-up criterion

$$T_{\max}<+\infty \text{ implies } \lim_{t\nearrow T_{\max}}\|\psi(t)\|_{H^1(\mathbb{R}^3)}=+\infty.$$

The sharp threshold of global existence and blowup for (1.1) with general Hartree-type $(|x|^{-\alpha}*|\psi|^2)\psi$ was considered in [10] for the case $N\geq 3, 2\leq\alpha<\min\{4,N\}, 1+\frac{4}{N}<p<\frac{N+2}{N-2}$ and in [11,12] for the case $N\geq 3, 2<\alpha<\min\{4,N\}, 1<p\leq 1+\frac{4}{N}$. The existence and multiplicity of normalized solutions for (1.1) was obtained in [13]. In particular, the existence and the non-degeneracy of ground state solutions were proved and further the minimal mass blowup solutions were constructed in [4] for (1.1) with $p=1+\frac{4}{3}$. However, to our knowledge, the stability and instability (especially strong instability) of standing waves for (1.1) has not been studied in the literature.

By a standing wave, we mean a solution to (1.1) with the form

$$\psi(t,x)=e^{i\omega t}u_\omega(x),\qquad \omega\in\mathbb{R},\ u_\omega(x)\in H^1(\mathbb{R}^3)\backslash\{0\},$$